Second Order Slip-factor

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Objectives

- At transition crossing the slip factor, \( \eta(p) = \alpha - 1/\gamma^2 \), becomes zero.
- Consequently, the spread of particle revolution frequencies approaches zero near transition

\[
\frac{\Delta f}{f} = \eta(p) \frac{\Delta p}{p} + \eta_p(p) \left( \frac{\Delta p}{p} \right)^2 + ...
\]

\( \Rightarrow \) At transition

\[
\sigma_f \equiv \left( \frac{\Delta f}{f} \right)^2 = \eta_p(p_{tr}) \sqrt{\left( \frac{\Delta p}{p} \right)^4} \quad \text{For Gaussian Distribution} \quad \sqrt{3} \eta_p(p_{tr}) \sigma_p^2
\]

- The second order slip-factor plays important role at the transition and good understanding of its behavior on machine parameters are important for both simulations and transition crossing tuning
- Initial computations using MAD showed that it does not deliver a reliable number
  - As result a study was initiated
    - Other participants: F. Schmidt - CERN, A. Valishev - FNAL
  - Also compared to two earlier publications
**Smooth Lattice Approximation**

- It is straightforward to get the exact analytical result in the smooth lattice approximation

\[ B_z = B_0 \left( 1 + g \frac{r}{R_0} + s \left( \frac{r}{R_0} \right)^2 + \ldots \right) \]

where the reference orbit is determined as \( \frac{1}{R_0} = \frac{eB_0}{p_0 c} \)

\[ \Rightarrow \quad p_0 c \left( 1 + \frac{\delta p}{p_0} \right) = eR_0 \left( 1 + \frac{r}{R_0} \right) B_0 \left( 1 + g \frac{r}{R_0} + s \left( \frac{r}{R_0} \right)^2 + \ldots \right) \Rightarrow \frac{\delta p}{p_0} = \frac{r}{R_0} (1 + g) + (g + s) \left( \frac{r}{R_0} \right)^2 + \ldots \]

- On the other hand the betatron frequency is determined by

\[ \nu_x^2 = 1 + \frac{R \, dB}{B \, dr} = 1 + g + \frac{r}{R_0} \left( g - g^2 + 2s \right) + \ldots \]

\[ \Rightarrow \quad \text{Betatron tune: } \nu_{x0} = \sqrt{1 + g} \quad \text{and Chromaticity } \xi \equiv p \frac{d\nu_x}{dp} = \frac{g - g^2 + 2s}{2
u_x^3} \]

- For the slip-factors we have:

\[ \alpha = \frac{p_0}{R_0} \frac{dr}{dp} = \frac{1}{1 + g} = \frac{1}{\nu_x^2} \]

\[ \alpha_p = \frac{p_0^2}{2R_0} \frac{d^2r}{dp^2} = - \frac{g + s}{(1 + g)^3} = - \frac{g + s}{\nu_x^2} \]

- That allows to bind up the chromaticity and the 2nd order slip-factor

\[ \alpha_p = \frac{1 - \nu_x^2 - 2\nu_x \xi}{2\nu_x^4} \]

- The 2nd order slip-factor is uniquely determined by the measurable values: the betatron tune and the tune chromaticity.
**FO Lattice (with rectangular dipoles and thin lenses)**

- It is a straightforward to obtain a semi-analytical solution in the case of one thin lens and one rectangular dipole per period
  - We expect that for a small betatron phase advance per cell the solution will coincide with smooth lattice approximation
  - We are interested to see the effect of beta-beating on the 2\textsuperscript{nd} order slip-factor

- The following lattice was used in simulations: $R_0=5$ m, $L_{dr}=0.3$ m
  - Number of periods and betatron phase advance were varied

![Graph showing BETA_X, BETA_Y, DISP_X, and DISP_Y over a range of 20 periods and 120 deg. per cell.](image)
Results of Computations for the FO Lattice

- Results of numeric computations show quite good coincidence with the smooth lattice model
  - Computations were done for 5, 10, 20 and 40 periods in the ring
  - All computations show behavior which is very close to the smooth lattice approximation
    - The reasons are not clear yet
  - It has been used to verify that MADX delivers correct result
    - Looks like for a knowledgeable person it does
  - Computations for the Booster and Main Injector will follow
FODO Lattice (with sector dipoles and thin lenses)

- Another simple example is FODO lattice with sector dipoles and no straights
- Exact Solution is obtained numerically
  - Iterations are used to find the trajectory locations in quads
- The following lattice was used in simulations: $R_0=5$ m
  - Number of periods and betatron phase advance were varied
Results of Computations for the FODO Lattice

- The same as for the FO lattice the results of computations show quite good coincidence with the smooth lattice model
  - Computations were done for 20 and 40 periods in the ring
  - All computations show that the slope is very close to the smooth lattice approximation
    - The reasons of offsets are not clear
  - Typically, zeroing of 2\textsuperscript{nd} order slip-factor happens for positive chromaticity in the range [0,10] for the phase advances of 45-90 deg. per cell
Results obtained in earlier studies

- Two earlier publications were also considered:
  
  

- Both papers
  
  - Written to understand transition crossing in the MI
  - Use sector dipoles and zero length quadrupoles
  - For zero sextupole strength and quadrupole focusing much larger than focusing coming from the dipoles their results (e.g. Eq. 5.8 in Ref. 2) coincide well with calculations presented above
    
    - Effect of sextupoles (and chromaticity) is not presented in sufficient details in Ref. [1]
    
    - Part of the equation describing effect of sextupoles on $\alpha_p$ in Ref [2] has incorrect dimension
**From 2nd order momentum compaction to 2nd order slip-factor**

\[
\frac{\Delta f}{f} = \eta(p) \frac{\Delta p}{p} + \eta_p(p) \left( \frac{\Delta p}{p} \right)^2 + \ldots
\]

\[
\eta(p) = \alpha - \frac{1}{\gamma^2}
\]

- Non-linear dependence of particle velocity on momentum should be also accounted

\[
\eta_p(p) = \alpha_p + \frac{3}{2} \frac{\beta^2}{\gamma^2} - \frac{\eta(p)}{\gamma^2}
\]
Relationship of 2\textsuperscript{nd} Momentum Compaction & Chromaticity

- **Kick from sextupole:**  \[ \Delta \theta = -\frac{e(SL)}{2pc} (D\delta)^2 \]

- **Let's find orbit change:**  
  \[
  \begin{bmatrix}
  x \\
  \theta - \Delta \theta
  \end{bmatrix} = \begin{bmatrix}
  M_{11} & M_{12} \\
  M_{21} & M_{22}
  \end{bmatrix}
  \begin{bmatrix}
  x \\
  \theta
  \end{bmatrix}
  \Rightarrow
  \begin{bmatrix}
  x \\
  \theta
  \end{bmatrix} = \frac{1}{2 - M_{11} - M_{22}} \begin{bmatrix}
  M_{21} & M_{12} \\
  M_{22} & M_{11}
  \end{bmatrix}
  \begin{bmatrix}
  0 \\
  \Delta \theta
  \end{bmatrix}
  \]

- **Corresponding orbit lengthening:**  \[ \Delta s = M_{51}x + M_{52}\theta \]

  \[
  M_{11} = c_v + \alpha s_v, \quad M_{12} = \beta_x s_v, \quad M_{21} = -(1 + \alpha^2) s_v / \beta_x, \quad M_{22} = c_v - \alpha s_v,
  \]

- **Using**  
  \[ M_{51} = D' \left(1 - c_v - \alpha s_v\right) - \frac{1 + \alpha^2}{\beta_x} Ds_v, \quad M_{52} = D \left(c_v - \alpha s_v - 1\right) - D' \beta_x s_v, \]

  \[
  \text{where} \quad c_x = \cos(2\pi v), \quad s_x = \sin(2\pi v), \quad D' = \frac{dD}{ds}, \quad \alpha = -\frac{1}{2} \frac{d\beta_x}{ds},
  \]

  one obtains:  \[ \Delta s = -\frac{e(SL)D^3\delta^2}{2pc} \Rightarrow \Delta \alpha_p \equiv \frac{1}{2C} \frac{d^2}{d\delta^2} \Delta s = -\frac{e(SL)D^3}{2pcC} \]

- **For many sextupoles we have:**  \[ \Delta \alpha_p = -\frac{e}{2pcC} \sum_n (SL)_n D_n^3 \]

- **While for the chromaticity we have:**  \[ \Delta \zeta = \frac{e}{4\pi pc} \sum_n (SL)_n D_n \beta_{\zeta n} \]

- **If all sextupoles are located at the same D's and \( \beta \)'s we obtain:**  \[
  \frac{d\alpha_p}{d\zeta} = -\frac{2\pi D^2}{C \beta_x} \quad \text{smooth optics appr.} \rightarrow \frac{1}{\nu_x^3}
  \]
Conclusions

- The main goal of this work is to verify that MADX delivers correct value for the second order slip-factor.
- As a by-product, we found that the 2nd order slip-factor and the horizontal chromaticity are related by simple equation:
  \[
  \frac{d\alpha_p}{d\xi} \approx -\frac{1}{\nu_x^3}
  \]
- For typical cases the 2nd order slip-factor at zero chromaticity is:
  \[
  |\alpha_p(\xi = 0)| \leq \frac{1}{\nu_x^2}
  \]